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Exact Solutions Fractional Heat-Like and Wave-Like Equations with Variable Coefficients

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Abstract

A relatively new analytical method, the homotopy decomposition method (HDM) was applied to derive exact and approximate analytical solutions of nonlinear Fractional heat-like and wave-like equations. In all examples, in the limit of infinitely many terms yields the exact solution. A comparison with the exact solution reveals that HDM is simple, efficient and reliable. In addition, the calculations involved in HDM are very simple and straight forward. It is demonstrated that HDM is a powerful and efficient tool for Fractional heat-like and wave-like equations. It was also demonstrated that HDM is more efficient than the ADM (Adomian decomposition method), VIM (Variational Iteration method), HAM (Homotopy analysis method) and HPM (Homotopy decomposition method).

Keywords: Fractional heat-like and wave-like equations; Homotopy decomposition method; Fractional derivative order

Introduction

Fractional Calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory signal, image processing and groundwater problems. In the past several decades, the investigation of travelling-wave solutions for nonlinear equations has played an important role in the study of nonlinear physical phenomena .An excellent literature of this can be found in fractional differentiation and integration operators were also used for extensions of the diffusion and wave equations [1-11].

The solutions of Fractional heat-like and wave-like equations with variable coefficients have attracted attention of many authors in mathematics community. Recently, Shou and He [12] used the variational iteration method (VIM) to solve various kinds of heat-like and wave-like equations. However with VIM one needs first to obtain, the Lagrange multiplier and the correctional function. In addition of this, sometime, the solutions obtained via the VIM are noisy [13,14] one therefore needs to cancel the noisy term to obtain the correct solution.

Xu and Cang [15] solved the fractional heat-like and wave-like equations with variable coefficients using Homotopy Analysis Method (HAM). The disadvantage of HAM is that, it is very much depended on choosing auxiliary parameter. Momani [13] applied the Adomian Decomposition method to the time fractional heat-like and wave-like equations with variable coefficients. The main disadvantage of the Adomian method is that the solution procedure for calculation of Adomian polynomials is complex and difficulty as pointed out by many researchers [16-20].

In this paper, we extend the application of the Homotopy Decomposition Method (HDM) in order to derive analytical approximate solutions to nonlinear time Fractional heat-like and wave-like equations with variable coefficients

The HDM was recently applied to solve: Fractional modified Kawahara equation, fractional model of HIV infection of CD4+T cells, attractor fractional one-dimensional Keller-Segel equations, fractional Jaulent-Miodek and Whitham-Broer-Kaup equations; Fractional Riccati Differential Equation, fractional nonlinear predator-prey population, fractional nonlinear system predator-prey population. The relatively new approached the HDM is a promising analytical technique to solve nonlinear fractional partial and ordinary differentials equations. The fractional partial differential equation under investigation here is given below as [21]:

$$\frac{\partial^{\alpha}u(x,y,z,t)}{\partial t^{\alpha}}=f\left(x,y,z\right)u_{x,x}+k\left(x,y,z\right)u_{y,y}+h\left(x,y,z\right)u_{zz},x,y,z\in I\subset\mathbb{R}_{*}^{+} \tag{1.1}$$

Subject to the initial conditions:

$$(x,y,z,0)=l(x,y,z),\ \partial_{t}u(x,y,z,0)=d(x,y,z) \tag{1.2}$$

The remaining of this paper is structured as follows: In section 2 we present a brief history of the fractional derivative order and theirs properties. We present the basic ideal of the homotopy decomposition method for solving high order nonlinear fractional partial differential equations. We present the application of the HDM for fractional nonlinear differential equations (1.1) and (1.2) and numerical results in Section 4. The conclusions are then given in the final Section 5.

Fractional Derivative Order

Brief history

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann–Liouville and the Caputo derivatives. For Caputo we have

$$ {}_{0}^{C}D_{x}^{\alpha}\left(f\left(x\right)\right)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\left(x-t\right)^{n-\alpha-1}\frac{d^{n}f(t}{dt^{n}})dt \tag{2.1}$$

For the case of Riemann-Liouville we have the following definition

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$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}f(t)dt \tag{2.2}$$

Each fractional derivative presents some advantages and disadvantages [22,23]. The Riemann–Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero but demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense [22]. Recently, Jumarie [21,22] proposed a simple alternative definition to the Riemann–Liouville derivative.

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1}\left\{f(t)-f(0)\right\}dt \tag{2.3}$$

His modified Riemann–Liouville derivative seems to have advantages of both the standard Riemann–Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non-differentiable) functions and the fractional derivative of a constant is equal to zero. However from it definition we do not actually give a fractional derivative of a function says $f(x)$ but the fractional derivative of $f(x)$-$f(0)$ and can sometime leads to fractional derivative that is not defined at the origin for some function [21].

Caputo and Riemann-Liouville may have their disadvantages, but they still remain the best definition of the fractional derivative. Every definition must be used accordingly [22].

Properties and definitions

Definition 1: A real function $f(x)$,x>0, is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number p>μ, such that $f(x)=x^p h(x)$, where $h(x)\in C[0,\infty])$, and it is said to be in space C_μ^m if $f(m)\in C_\mu$, $m\in\mathbb{N}$

Definition 2: The Riemann-Liouville fractional integral operator of order α ≥ 0, of a function $f\in C_\mu$, μ ≥ -1, is defined as

$$J^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\alpha>0,x>0 \tag{2.4}$$

$J^0 f(x)=f(x)$

Properties of the operator can be found in [22] we mention only the following:

For $f\in C_\mu$,$\mu\geq -1$,$\alpha,\beta\geq 0$ and $\gamma>-1$: (2.5)

$J^\alpha J^\beta f(x)=J^{(\alpha+\beta)}f(x)$, $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$ and $J^{\alpha}x^{\gamma}=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$

Lemma 1: If $m-1<\alpha\leq m, m\in\mathbb{N}$ *and* $f\in C_\mu^m,\mu\geq-1$, then

$$D^{\alpha}J^{\alpha}f(x)=f(x)\text{ and, }J^{\alpha}D_0^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}\left(0^+\right)\frac{x^k}{k!},x>0 \tag{2.6}$$

Definition 3: Partial Derivatives of Fractional order

Assume now that $f(x)$ is a function of n variables x_i i=1,……, n also of class C on $D\in\mathbb{R}_n$. As an extension of definition 3 we define partial derivative of order α for f respect to x_i the function

$$a\partial_{\underline{x}}^{\alpha}f=\frac{1}{\Gamma(m-\alpha)}\int_a^{x_i}(x_i-t)^{m-\alpha-1}\partial_{x_i}^m f(x_j)|_{x_j=t}\,dt \tag{2.7}$$

If it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

Basic Idea of the HDM

To illustrate the basic idea of this method we consider a general nonlinear non-homogeneous fractional partial differential equation with initial conditions of the following form

$$\frac{\partial^{\alpha}U(x,t)}{\partial t^{\alpha}}=L\left(U(x,t)\right)+N\left(U(x,t)\right)+f(x,t),\alpha>0 \tag{3.1}$$

Subject to the initial condition

$D_0^{\alpha-k}U(x,0)=f_k(x),(k=0,\ldots\ldots n-1),D_0^{\alpha-n}U(x,0)=0$ and $n=[\alpha]$

$D_0^{k}U(x,0)=g_k(x),(k=0,\ldots\ldots n-1),D_0^{n}U(x,0)=0$ and $n=[\alpha]$

Where, $\dfrac{\partial^\alpha}{\partial t^\alpha}$ denotes the Caputo or Riemann-Liouville fraction derivative operator, f is a known function, N is the general nonlinear fractional differential operator and L represents a linear fractional differential operator. The method first step here is to transform the fractional partial differential equation to the fractional partial integral equation by applying the inverse operator $\dfrac{\partial^\alpha}{\partial t^\alpha}$ of on both sides of equation (3.1) to obtain: In the case of Riemann-Liouville fractional derivative

$$U(x,t)=\sum_{j=0}^{n-1}\frac{f_j(x)}{\Gamma(\alpha-j+1)}t^{\alpha-j}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[L(U(x,\tau))+N(U(x,\tau))+f(x,\tau)\right]d\tau \tag{3.2}$$

In the case of Caputo fractional derivative

$$U(x,t)=\sum_{j=0}^{n-1}\frac{g_j(x)}{\Gamma(\alpha-j+1)}t^{j}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[L(U(x,\tau))+N(U(x,\tau))+f(x,\tau)\right]d\tau$$

Or in general by putting

$$\sum_{j=0}^{n-1}\frac{f_j(x)}{\Gamma(\alpha-j+1)}t^{\alpha-j}=f(x,t)\quad\text{or}\quad f(x,t)=\sum_{j=0}^{n-1}\frac{g_j(x)}{\Gamma(\alpha-j+1)}t^{j}$$

We obtain:

$$U(x,t)=T(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left[L(U(x,\tau))+N(U(x,\tau))+f(x,\tau)\right]d\tau \tag{3.3}$$

In the homotopy decomposition method, the basic assumption is that the solutions can be written as a power series in p

$$U(x,t,p)=\sum_{n=0}^{\infty}p^nU_n(x,t), \tag{3.4 a}$$

$$U(x,t)=\lim_{p\to 1}U(x,t,p) \tag{3.4 b}$$

and the nonlinear term can be decomposed as

$$NU(x,t)=\sum_{n=0}^{\infty}p^n\mathcal{H}_n(U) \tag{3.5}$$

Where $p\epsilon(0,1)$ is an embedding parameter. $\mathcal{H}_n(U)$ (U) is the He's polynomials that can be generated by

$$\mathcal{H}_n(U_0,\cdots\cdots,U_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{j=0}^{\infty}p^jU_j(x,t)\right)\right],n=0,1,2\cdots\cdots \tag{3.6}$$

The homotopy decomposition method is obtained by the graceful coupling of homotopy technique with Abel integral and is given by

$$\sum_{n=0}^{\infty}p^nU_n(x,t)-T(x,t)=\frac{p}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}$$

$$\left[f(x,\tau) + L\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right) + N\left(\sum_{n=0}^{\infty} p^n U_n(x,\tau)\right)\right] d\tau \quad (3.7)$$

Comparing the terms of same powers of p gives solutions of various orders with the first term:

$$U_0(x,t) = T(x,t) \quad (3.8)$$

Complexity and convergence of the homotopy decomposition method

It is very important to test the computational complexity of a method or algorithm. Complexity of an algorithm is the study of how long a program will take to run, depending on the size of its input and long of loops made inside the code. We compute a numerical example which is solved by the homotopy decomposition method. The code has been presented with Mathematica 8 according to the following code.

Step 1: Set $m \leftarrow 0$

Step 2: Calculated the recursive relation after the comparison of the terms of the same power is done.

Step 3: If $\| U_{n+1}(x,t) - U_n(x,t) \| < r$ with r the ratio of the neighbourhood of the exact solution [2] then go to step 4, else $m \leftarrow m+1$ and go to step 2.

Step 4: Print out:

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t)$$

as the approximate of the exact solution.

Lemma 1: If the exact solution of the fractional partial differential equation (3.1) exists, then

$$\| U_{n+1}(x,t) - U_n(x,t) \| < r \text{ for all } (x,t) \in X \times T$$

Proof: Let $(x,t) \in X \times T$, then since the exact solution exists, then we have that following:

$$\| U_{n+1}(x,t) - U_n(x,t) \| = \| U_{n+1}(x,t) - U(x,t) + U(x,t) - U_n(x,t) \|$$

$$\le \| U_{n+1}(x,t) - U(x,t) \| + \| -U_n(x,t) + U(x,t) \|$$

$$\le \frac{r}{2} + \frac{r}{2} = r$$

The last inequality follows from [21].

Lemma 2: The complexity of the homotopy decomposition method is of order $O(n)$

Proof: The number of computations including product, addition, subtraction and division are

In step 2

U_0: 0 because, obtains directly from the initial guess [23]

U_1: 3

⋮

U_n: 3

Now in step 4 the total number of computations is equal to $\Sigma_{j=0}^{n} U_j(x,t) = 3n = O(n)$.

Theorem 1 [23]: Assuming that XxT $\subset R \times R^+$ is a Banach space with a well defined norm $\| \ \|$, over which the series sequence of the approximate solution of (1.1) is defined, and the operator $G(U_n(x,t)) = U_{n+1}(x,t)$ defining the series solution of (1.4b) satisfies the Lipschitzian conditions that is $\| G(U_k^*) - G(U_k) \| \le \varepsilon \| U_k^*(x,t) - U_k(x,t) \|$ for all $(x,t,k) \in X \times T \times \mathbb{N}$, then series solution obtained (1.5) is unique.

Proof: Assume that $U(x,t)$ and $U^*(x,t)$ is the series solution satisfying equation (1.1) then:

$$U^*(x,t,p) = \sum_{n=0}^{\infty} p^n U_n^*(x,t) \text{ with initial guess } T(x,t)$$

$$U(x,t,p) = \sum_{n=0}^{\infty} p^n U_n(x,t) \text{ also with initial guess } T(x,t) \text{ therefore}$$

$$\| U_n^*(x,t) - U_n(x,t) \| = 0, n = 0,1,2,\cdots\cdots\cdots$$

By the recurrence for $n = 0, U_n^*(x,t) = U_n(x,t) = T(x,t)$, assume that for n>k ≥ 0, $\| U_k^*(x,t) - U_k(x,t) \| = 0$. Then

$$\| U_{k+1}^*(x,t) - U_{k+1}(x,t) \| = \| G(U_k^*) - G(U_k) \| \le \varepsilon \| U_k^*(x,t) - U_k(x,t) \| = 0$$

This completes the proof.

Application

In learning science examples are useful than rules" (Isaac Newton). In this section we apply this method for solving fractional differential equation in form of equation (1.1) together with (1.2).

Example 1: Consider the following three-dimensional fractional heat-like equation

$$\partial_t^\alpha u(x,y,z,t) = x^4 y^4 z^4 + \frac{1}{36}\left(x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz}\right), 0 < x,y,z < 1, 0 < \alpha \le 1 \quad (4.1)$$

Subject to the initial condition:

$$u(x,y,z,0) = 0 \quad (4.2)$$

Following carefully the steps involved in the HDM, we arrive at the following equations

$$\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) = \frac{p}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(x^4 y^4 z^4 + \frac{1}{36} \left(\begin{array}{l} x^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{xx} \\ + y^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{yy} + z^2 \left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \right)_{zz} \end{array} \right) \right) d\tau \quad (4.3)$$

Now comparing the terms of the same power of p yields:

$$p^0 : u_0(x,y,z,t) \quad (4.4)$$

$$p^1 : u_1(x,y,z,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x^4 y^4 z^4 d\tau$$

$$p^n : u_n(x,y,z,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{1}{36} \left(x^2 (u_{n-1})_{xx} + y^2 (u_{n-1})_{yy} + z^2 (u_{n-1})_{zz} \right) \right) d\tau, u_n(x,y,z,0) = 0, n \ge 2$$

Thus the following components are obtained as results of the above integrals

$u_0(x,y,z,t) = 0$

$$u_1(x,y,z,t) = \frac{t^\alpha x^4 y^4 z^4}{\Gamma(\alpha+1)}$$

$$u_2(x,y,z,t) = \frac{t^{2\alpha} x^4 y^4 z^4}{\Gamma(2\alpha+1)}$$

$$u_3(x,y,z,t) = \frac{t^{3\alpha} x^4 y^4 z^4}{\Gamma(3\alpha+1)}$$

⋮

$$u_n(x,y,z,t) = \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x,y,z,t) = \sum_{n=1}^{N} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)} \quad (4.6)$$

Now when $N \to \infty$ we obtained the follow solution

$$u(x,y,z,t) = \sum_{n=0}^{\infty} \frac{t^{n\alpha} x^4 y^4 z^4}{\Gamma(n\alpha+1)} - x^4 y^4 z^4 = x^4 y^4 z^4 \left(E_\alpha(t^\alpha) - 1 \right)$$

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Where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case α=1

$$u(x,y,z,t) = x^4y^4z^4(\exp(t)-1) \tag{4.7}$$

This is the exact solution for this case.

Example 2: we consider the three-dimensional fractional wave-like equation:

$$\partial_t^\alpha u(x,y,z,t) = x^2+y^2+z^2+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}), 0<x,y,z<1, 1<\alpha\le 2$$

Subject to the initial condition:

$$u(x,y,z,0)=0, u_t(x,y,z,0)=x^2+y^2-z^2 \tag{4.8}$$

Following carefully the steps involved in the HDM, we arrive at the following series solutions:

$$u_0(x,y,z,t)=(x^2+y^2-z^2)t$$

$$u_1(x,y,z,t)=\frac{t^\alpha}{\Gamma(1+\alpha)}(x^2+y^2-z^2)$$

$$u_2(x,y,z,t)=\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}(x^2+y^2+z^2)$$

$$u_3(x,y,z,t)=\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}(x^2+y^2-z^2)$$

$$\vdots$$

$$u_n(x,y,z,t)=\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}(x^2+y^2+(-1)^n z^2)$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x,y,z,t)=\sum_{n=1}^{N}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}(x^2+y^2+(-1)^n z^2) \tag{4.9}$$

Now when $N\to\infty$ we obtained the follow solution

$$u(x,y,z,t)=\sum_{n=1}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}(x^2+y^2+(-1)^n z^2) \tag{4.10}$$

In the case of α=2 we obtain:

$$u(x,y,z,t)=(x^2+y^2)\exp(t)+z^2\exp(-t)-(x^2+y^2+z^2)$$

This is the exact solution for this case.

Example 3: we consider the one-dimensional fractional wave-like equation:

$$\partial_t^\alpha u(x,t)=\frac{1}{2}x^2u_{xx},\ \ 0<x<1,\ 1<\alpha\le 2, t>0 \tag{4.11}$$

With the initial conditions as

$u(x,0)=x^2$

Following carefully the steps involved in the HDM, we arrive at the following series solutions:

$$u_0(x,t)=x^2$$

$$u_1(x,t)=\frac{t^\alpha x^2}{\Gamma(\alpha+1)}$$

$$u_2(x,t)=\frac{t^{2\alpha} x^2}{\Gamma(2\alpha+1)}$$

$$u_3(x,t)=\frac{t^{3\alpha} x^2}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u_n(x,t)=\frac{t^{n\alpha} x^2}{\Gamma(n\alpha+1)}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x,t)=\sum_{n=1}^{N}\frac{t^{n\alpha}x^2}{\Gamma(n\alpha+1)} \tag{4.12}$$

Now when $N\to\infty$ we obtained the follow solution

$$u(x,t)=\sum_{n=0}^{\infty}\frac{t^{n\alpha}x^2}{\Gamma(n\alpha+1)}=x^2E_\alpha(t^\alpha)$$

Where $E_\alpha(t^\alpha)$ is the generalized Mittag-Leffler function. Note that in the case α=1

$u(x,t)=x^2\exp(t)$

This is the exact solution for this case.

Example 4: In this example we consider the two-dimensional fractional heat-like equation

$$\partial_t^\alpha u(x,t)=u_{xx}+u_{yy}, 0<x,y<2\pi, t>0, 0<\alpha\le 1 \tag{4.13}$$

Subject to the initial condition:

$$u(x,y,0)=\sin(x)\sin(y) \tag{4.14}$$

Following carefully the steps involved in the HDM, we arrive at the following series solutions:

$$u_0(x,y,t)=\sin(x)\sin(y)$$

$$u_1(x,y,t)=-2\frac{t^\alpha\sin(x)\sin(y)}{\Gamma(\alpha+1)}$$

$$u_2(x,y,t)=4\frac{t^{2\alpha}\sin(x)\sin(y)}{\Gamma(2\alpha+1)}$$

$$u_3(x,y,t)=-8\frac{t^{3\alpha}\sin(x)\sin(y)}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u_n(x,y,z,t)=(-2)^n\frac{t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)}$$

Therefore the approximate solution of equation for the first n is given below as:

$$u_N(x,y,t)=\sum_{n=1}^{N}(-2)^n\frac{t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)} \tag{4.12}$$

Now when $N\to\infty$ we obtained the follow solution

$$u(x,y,t)=\sum_{n=0}^{\infty}\frac{(-2)^n t^{n\alpha}\sin(x)\sin(y)}{\Gamma(n\alpha+1)}$$

Note that in the case α=1

$u(x,y,z,t)=\sin(x)\sin(y)\exp(-2t)$

This is the exact solution for this case.

Conclusion

We derived approximated solutions of Fractional heat-like and wave-like equations with variable coefficients using the relatively new analytical technique the HDM. We presented the brief history and some properties of fractional derivative concept. It is demonstrated that HDM is a powerful and efficient tool of FPDEs. In addition, the calculations involved in HDM are very simple and straightforward. Comparing the methodology HDM to HPM, ADM, VIM and HAM have the advantages. Disparate the ADM, the HDM is free from the need to use Adomian polynomials. In this method we do not need the Lagrange multiplier, correction functional, stationary conditions, or calculating heavy integrals, the solution obtained are noise free, which eliminate the complications that exist in the VIM. In contrast to the HAM, this method is not required to solve the functional equations in iteration each the efficiency of HAM is very much depended on choosing auxiliary parameter. In contract to HPM, we do not need to continuously deform a difficult problem to another that is easier to solve. We can easily conclude that the Homotopy Decomposition method is an efficient tool to solve approximate solution of nonlinear fractional partial differential equations.

References

1. Oldham KB, Spanier J (1974) The Fractional Calculus: Theory and Applications

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of Differentiation and Integration to Arbitrary Order (Dover Books on Mathematics). Academic Press, New York, NY, USA.

2. Podlubny I (1999) Fractional Differential Equations, Academic Press, New York, NY, USA.
3. Kilbas AA, Srivastava HH, Trujillo JJ (2006) Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, The Netherlands.
4. Caputo M (1967) Linear Models of Dissipation whose Q is almost Frequency Independent-II. Geophysical Journal International 13: 529-539.
5. Miller KS, Ross B (1993) An Introduction to the Fractional Calculus and Fractional Differential Equations. Jhn Wiley & Son, New York, USA, 384.
6. Samko SG, Kilbas AA, Marichev OI (1993) Fractional Integrals and Derivatives: Theory and Applications, Taylor & Francis Group, Yverdon, Switzerland 976.
7. Zaslavsky GM (2005) Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, Oxford, USA, 448.
8. Atangana A (2012) Numerical solution of space-time fractional derivative of groundwater flow equation, International conference of algebra and applied analysis, June 20-24, Istanbul, 20.
9. Schneider WR, Wyss W (1989) Fractional diffusion and wave equations. J Math Phys 30: 134-144.
10. Atangana A (2012) New Class of Boundary Value Problems. Inf Sci Lett 2: 67-76.
11. Odibat ZM, Momani S (2008) Application of variational iteration method to nonlinear equations of fractional order. Int J Nonlinear Sci Numer Simul 1: 15.
12. Shou DH, He JH (2008) Beyond Adomain method: The variational iteration method for solving heat like and wave-like equations with variable coefficients. Physics Letters A 372: 233-237.
13. Momani Shaher (2005) Analytical approximate solution for fractional heat-like and wave-like equations with variable coefficients using the decomposition method. Appl Math Comput 165: 459-472.
14. Bildik N, Konuralp A (2006) The Use of Variational Iteration Method, Differential Transform Method and Adomian Decomposition Method for Solving Different Types of Nonlinear Partial Differential Equations. Int J Nonlin- ear Sci Numer Simul 7: 65-70.
15. Xu H, Cang J (2008) Analysis of a time fractional wave-like equation with the homotopy analysis method. Phys Lett A 372: 1250-1255.
16. Sweilam NH, Khader MM (2007) Variational iteration method for one dimensional nonlinear thermoelasticity. Chaos Solitons and Fractals. 32: 145-149.
17. Momani S, Odibat Z (2007) Numerical comparison of methods for solving linear differential equations of fractional order. Chaos, Solitons & Fractals 31: 1248-1255.
18. Soliman AA (2006) A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations. Chaos, Solitons and fractals 29: 294-302.
19. Miller KS, Ross B (1993) An introduction to the fractional calculus and fractional differential equations, Wiley, New York, USA, pp. 384.
20. Samko SG, Kilbas AA, Marichev OI (1993) Fractional integrals and derivatives, Translated from the 1987 Russian original, Gordon and Breach.
21. Jumarie G (2005) On the representation of fractional Brownian motion as an integral with respect to (dt)a. Appl Math Lett 18: 739-748.
22. Jumarie G (2006) Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Comput Math Appl 51: 1367-1376.
23. Atangana A (2012) Homotopy decomposition for solving high order fractional nonlinear partial differential equation and its convergence. Paper submitted for publication in Applied Mathematics and Computation.